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A Characterization of Depth 2 Subfactors of II_1 Factors

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We characterize finite index depth 2 inclusions of type II_1 factors in terms of actions of weak Kac algebras and weak C^*-Hopf algebras. If $N \subset M \subset M_1 \subset M_2 \subset \cdots$ is the Jones tower constructed from such an inclusion $N \subset M$, then $B = M' \cap M_2$ has a natural structure of a weak C^*-Hopf algebra and there is a minimal action of B on M_1 such that M is the fixed point subalgebra of M_1 and M_2 is isomorphic to the crossed product of M_1 and B. This extends the well-known results for irreducible depth 2 inclusions. © 2000 Academic Press

1. INTRODUCTION

Let $N \subset M$ be a finite index depth 2 inclusion of type II_1 factors and $N \subset M \subset M_1 \subset M_2 \subset \cdots$ the corresponding Jones tower. It was announced by A. Ocneanu and was proved in [23, 4, 13] that if $N \subset M$ is irreducible, i.e., such that $N' \cap M = \mathbb{C}$, then $B = M' \cap M_2$ has a natural structure of a finite-dimensional Kac algebra and there is a canonical outer action of B on M_1 such that $M = M_1^B$, the fixed point subalgebra of M_1 with respect to this action, and M_2 is isomorphic to the crossed product $M_1 \rtimes B$. The outerness condition is equivalent to the relative commutant $M_1' \cap M_1 \rtimes B$ being trivial (such actions are also called minimal). In the case of an infinite index a similar description in terms of multiplicative unitaries and quantum groups was obtained in [5].

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In this work we extend the above result to (in general, reducible, i.e., such that $\mathbb{C} \subset N' \cap M$) finite index depth 2 inclusions of type II_1 factors. We replace usual Kac algebras (Hopf C^*-algebras) by weak Kac algebras [14] or weak C^*-Hopf algebras [2]. A weak Kac algebra is a special case of a weak C^*-Hopf algebra characterized by the property $S^2 = \mathrm{id}$. It was shown in [14] that the category of weak Kac algebras is equivalent to those of generalized Kac algebras of T. Yamanouchi [27] (another proof of that can be found in [18]) and of Kac bimodules (an algebraic version of Hopf bimodules of J.-M. Vallin [25]). The advantage of the language of weak Kac algebras and weak C^*-Hopf algebras is that their defining axioms are clearly self-dual, so it is easy to work with both weak Kac algebra (weak C^*-Hopf algebra) and its dual simultaneously.

Let us mention that a possibility of characterizing finite index depth 2 inclusions in terms of weak C^*-Hopf algebras was suggested in [17]. For an arbitrary (possibly infinite) index M. Enock and J.-M. Vallin have obtained a similar description in terms of pseudo-multiplicative unitaries [7].

The paper is organized as follows.

In Section 2 (Preliminaries) we briefly review, following [14, 2] and [17], the basic definitions and facts of the theory of weak Kac algebras and weak C^*-Hopf algebras, including their actions of von Neumann algebras.

Section 3 is devoted to establishing a non-degenerate duality between the finite dimensional C^*-algebras $A = N' \cap M_1$ and $B = M' \cap M_2$, which gives a natural coalgebra structures on them.

In Sections 4 and 5 we investigate the relations between algebra and coalgebra structures on B, following the general strategy of Szymanski's reasoning [23] based on the above duality. It turns out that the square of the corresponding antipode is implemented by a positive invertible element determined by Index $\tau|_{M' \cap M_1}$, the Watatani index [26] of the restriction of the Markov trace τ on $M' \cap M_1$. That is why it is natural to consider the cases of scalar and non-scalar Index $\tau|_{M' \cap M_1}$ in which the antipode is respectively involutive and non-involutive. The main result here is that in the mentioned cases B and A are biconnected weak Kac algebras and weak C^*-Hopf algebras respectively (they are usual Kac algebras iff the inclusion $N \subset M$ is irreducible). We also prove in Section 4, that if $[M : N]$ is an integer which has no divisors of the form n^2, $n > 1$, then the inclusion is irreducible and B is a Kac algebra acting outerly on M_1. In particular, if $[M : N] = p$ is prime, then B must be the group algebra of the cyclic group $G = \mathbb{Z}/p\mathbb{Z}$.

In Section 6 we show that there exists a canonical (left) minimal action of B on M_1 such that M is the fixed point subalgebra of M_1 with respect to this action, and M_2 is isomorphic to $M_1 \rtimes B$, the crossed product of M_1 and B. The minimality condition means that the relative commutant

$M_1' \cap M_1 \rtimes B$ is minimal possible, in which case it is isomorphic to the Cartan subalgebra $B_s \subset B$.

It is important to stress that in the above situation one can take

$$\begin{array}{ccc} B^* & \subset & B^* \rtimes B \\ \cup & & \cup \\ B^* \cap B & \subset & B, \end{array}$$

where $B^* = A$, as a canonical commuting square [22] of the inclusion $M_1 \subset M_2$. The above square, and thus the equivalence class of inclusions, is completely determined by B. This implies that every biconnected weak C^*-Hopf algebra has at most one minimal action on a given II_1 factor and thus correspond to no more than one (up to equivalence) finite index depth 2 subfactor. Note that any biconnected weak Kac algebra admits a unique minimal action on the hyperfinite II_1 factor [16].

Finally, in Section 7 we explicitly describe biconnected weak Kac algebras corresponding to all non-isomorphic reducible depth 2 index 4 subfactors of the hyperfinite II_1 factor. We also give an example of a biconnected weak Hopf C^*-algebra of non-integer index $16\cos^4(\pi/5)$ constructed from the subfactor with the principal graph A_3.

Let us remark that this characterization of depth 2 inclusions means that weak Kac algebras provide a good setting for studying actions of usual Kac algebras on II_1 factors, since any (not necessarily minimal) action of a Kac algebra produces a depth 2 inclusion of von Neumann algebras and one can canonically associate with this action a weak Kac algebra completely describing it. More details on this will be published elsewhere.

2. PRELIMINARIES

Our main references to finite dimensional weak C^*-Hopf algebras are [2] and [18]. Weak Kac algebras, a special case of this notion characterized by the property $S^2 = \mathrm{id}$, were considered in [14]. These objects generalize both finite groupoid algebras and usual Kac algebras.

A *weak Kac algebra* B is a finite dimensional C^*-algebra equipped with the *comultiplication* $\Delta: B \to B \otimes B$, *counit* $\varepsilon: B \to \mathbb{C}$, and *antipode* $S: B \to B$, such that (Δ, ε) defines a coalgebra structure on B and the following axioms hold for all $b, c \in B$ (we use Sweedler's notation $\Delta(b) = b_{(1)} \otimes b_{(2)}$ for the comultiplication):

(1) Δ is a $*$-preserving (but not necessarily unital) homomorphism:

$$\Delta(bc) = \Delta(b)\,\Delta(c), \qquad \Delta(b^*) = \Delta(b)^{*\otimes *},$$

(2) The target counital map ε^t, defined by $\varepsilon^t(b) = \varepsilon(1_{(1)} b)\, 1_{(2)}$, satisfies the relations

$$b\varepsilon^t(c) = \varepsilon(b_{(1)} c)\, b_{(2)}, \qquad b_{(1)} \otimes \varepsilon^t(b_{(2)}) = 1_{(1)} b \otimes 1_{(2)},$$

(3) S is an anti-algebra and anti-coalgebra map such that $S^2 = \mathrm{id}$, $(S \circ *) = (* \circ S)$, and

$$b_{(1)} S(b_{(2)}) = \varepsilon^t(b).$$

If instead of the conditions $S^2 = \mathrm{id}$ and $(S \circ *) = (* \circ S)$ we have a less restrictive property $(S \circ *)^2 = \mathrm{id}$, then B is called a *weak C^*-Hopf algebra.*

Note that the axioms (2) and (3) above are equivalent to the following axioms for the source counital map $\varepsilon^s(b) = 1_{(1)} \varepsilon(b 1_{(2)})$:

(2′) $\varepsilon^s(c)\, b = b_{(1)} \varepsilon(c b_{(2)})$, $\varepsilon^s(b_{(1)}) \otimes b_{(2)} = 1_{(1)} \otimes b 1_{(2)}$,

(3′) $S(b_{(1)})\, b_{(2)} = \varepsilon^s(b)$.

The dual vector space B^* has a natural structure of a weak Kac algebra (weak C^*-Hopf algebra) given by dualizing the structure operations of B, see [2, 14].

The main difference between weak Kac (C^*-Hopf) algebras and classical Kac algebras is that the images of the counital maps are, in general, nontrivial unital C^*-subalgebras of B, called *Cartan subalgebras* (note that we have $\varepsilon^t \circ \varepsilon^t = \varepsilon^t$ and $\varepsilon^s \circ \varepsilon^s = \varepsilon^s$):

$$B_t = \{x \in B \mid \varepsilon^t(x) = x\} = \{x \in B \mid \Delta(x) = x 1_{(1)} \otimes 1_{(2)} = 1_{(1)} x \otimes 1_{(2)}\},$$
$$B_s = \{x \in B \mid \varepsilon^s(x) = x\} = \{x \in B \mid \Delta(x) = 1_{(1)} \otimes x 1_{(2)} = 1_{(1)} \otimes 1_{(2)} x\}.$$

The Cartan subalgebras commute: $[B_t, B_s] = 0$, also we have $S \circ \varepsilon^s = \varepsilon^t \circ S$ and $S(B_t) = B_s$. We say that B is *connected* [16] if $B_t \cap Z(B) = \mathbb{C}$ (where $Z(B)$ denotes the center of B), i.e., if the inclusion $B_t \subset B$ is connected. B is connected iff $B_t^* \cap B_s^* = \mathbb{C}$ ([16], Proposition 3.11). We say that B is *biconnected* if both B and B^* are connected.

Weak Kac (C^*-Hopf) algebras have integrals in the following sense.

There exists a unique projection $p \in B$, called a *Haar projection*, such that for all $x \in B$:

$$xp = \varepsilon^t(x)\, p, \qquad S(p) = p, \quad \varepsilon^t(p) = 1.$$

There exists a unique positive functional ϕ on B, called a *normalized Haar functional* (which is a trace iff B is a weak Kac algebra), such that

$$(\mathrm{id} \otimes \phi)\, \Delta = (\varepsilon^t \otimes \phi)\, \Delta, \qquad \phi \circ S = S, \qquad \phi \circ \varepsilon^t = \varepsilon.$$

The following notions of action, crossed product, and fixed point subalgebra were introduced in [17].

A (left) *action* of a weak Kac (C^*-Hopf algebra) B on a von Neumann algebra M is a linear map

$$B \otimes M \ni b \otimes x \mapsto (b \triangleright x) \in M$$

defining a structure of a left B-module on M such that for all $b \in B$ the map $b \otimes x \mapsto (b \triangleright x)$ is weakly continuous and

(1) $b \triangleright xy = (b_{(1)} \triangleright x)(b_{(2)} \triangleright y)$,

(2) $(b \triangleright x)^* = S(b)^* \triangleright x^*$,

(3) $b \triangleright 1 = \varepsilon^t(b) \triangleright 1$, and $b \triangleright 1 = 0$ iff $\varepsilon^t(b) = 0$.

A *crossed product* algebra $M \rtimes B$ is constructed as follows. As a $\mathbb{C}$-vector space it is $M \otimes_{B_t} B$, where B is a left B_t-module via multiplication and M is a right B_t-module via multiplication by the image of B_t under $z \mapsto (z \triangleright 1)$; that is, we identify

$$x(z \triangleright 1) \otimes b \equiv x \otimes zb$$

for all $x \in M$, $b \in B$, $z \in B_t$. Let $[x \otimes b]$ denote the class of $x \otimes b$. A $*$-algebra structure on $M \rtimes B$ is defined by

$$[x \otimes b][y \otimes c] = [x(b_{(1)} \triangleright y) \otimes b_{(2)} c],$$

$$[x \otimes b]^* = [(b^*_{(1)} \triangleright x^*) \otimes b^*_{(2)}]$$

for all $x, y \in A$, $b, c \in B$. It is possible to show that this abstractly defined $*$-algebra $M \rtimes B$ is $*$-isomorphic to a weakly closed algebra of operators on some Hilbert space [17], i.e., $M \rtimes B$ is a von Neumann algebra.

The collection $M^B = \{x \in M \mid b \triangleright x = \varepsilon^t(b) \triangleright x, \forall b \in B\}$ is a von Neumann subalgebra of M, called a *fixed point subalgebra.*

The relative commutant $M' \cap M \rtimes B$ always contains a $*$-subalgebra isomorphic to B_s. Indeed, if $z \in B_s$, then it follows easily from the axioms of a weak C^*-Hopf algebra that $\Delta(z) = 1_{(1)} \otimes 1_{(2)} z$, therefore

$$[1 \otimes z][x \otimes 1] = [(z_{(1)} \triangleright x) \otimes z_{(2)}] = [(1_{(1)} \triangleright x) \otimes 1_{(2)} z]$$

$$= [x \otimes z] = [x \otimes 1][1 \otimes z],$$

for all $x \in M$, and $B_s \subset M' \cap M \rtimes B$. We say that the action $\triangleright$ is *minimal* if $B_s = M' \cap M \rtimes B$.

3. DUALITY BETWEEN RELATIVE COMMUTANTS

Let $N \subset M$ be a depth 2 inclusion of type II_1 factors with a finite index $[M:N]=\lambda^{-1}$ and

$$N \subset M \subset M_1 \subset M_2 \subset \cdots$$

be the corresponding Jones tower, $M_1 = \langle M, e_1 \rangle$, $M_2 = \langle M_1, e_2 \rangle$, ..., where $e_1 \in N' \cap M_1$, $e_2 \in M' \cap M_2$, ... are the Jones projections. The depth 2 condition means that $N' \cap M_2$ is the basic construction of the inclusion $N' \cap M \subset N' \cap M_1$. Let τ be the normalized (Markov) trace on M_2.

With respect to this trace, the square of algebras in the upper right corner of the diagram below

$$\begin{array}{ccccc} N' \cap M & \subset & N' \cap M_1 & \subset & N' \cap M_2 \\ & & \cup & & \cup \\ & & M' \cap M_1 & \subset & M' \cap M_2 \\ & & & & \cup \\ & & & & M_1' \cap M_2 . \end{array}$$

is commuting ($E_{M_1} \circ E_{M'} = E_{M'} \circ E_{M_1}$ on $N' \cap M_2$) and non-degenerate, i.e., $N' \cap M_2 = (N' \cap M_1)(M' \cap M_2)$. This square is called a standard (or canonical) commuting square of the inclusion $M_1 \subset M_2$ [22].

Let us denote

$$\begin{aligned} &A = N' \cap M_1, && B = M' \cap M_2, \\ &A_t = N' \cap M, && A_s = M' \cap M_1 = B_t, \quad B_s = M_1' \cap M_2 . \end{aligned}$$

Note that A_t commutes with B, B_s commutes with A, and $A \cap B = A_s = B_t$.

The next lemma will be frequently used in the sequel without specific reference.

Lemma 3.1. *$(N' \cap M_2)\, e_2 = Ae_2$ and $(N' \cap M_2)\, e_1 = Be_1$. More precisely for any $x \in N' \cap M_2$ we have*

$$xe_2 = \lambda^{-1} E_{M_1}(xe_2)\, e_2, \qquad xe_1 = \lambda^{-1} E_{M'}(xe_1)\, e_1 .$$

Proof. This statement is a special case of ([19], Lemma 1.2) since $N' \cap M_2$ is the basic construction for the inclusions $N' \cap M \subset N' \cap M_1$ and $M_1' \cap M_2 \subset M' \cap M_2$ with the corresponding Jones projections e_2 and e_1 respectively.

Let us denote $d = \dim(M' \cap M_1)$.

PROPOSITION 3.2. *The form*

$$\langle a, b\rangle = d\lambda^{-2}\tau(ae_2e_1b), \qquad a\in A,\, b\in B$$

defines a non-degenerate duality between A and B.

Proof. If $a\in A$ is such that $\langle a, B\rangle = 0$, then

$$\tau(ae_2e_1B) = \tau(ae_2e_1(N'\cap M_2)) = 0,$$

therefore, using the Markov property of τ and properties of Jones projections, we get

$$\tau(aa^*) = \lambda^{-1}\tau(ae_2a^*) = \lambda^{-2}\tau(ae_2e_1(e_2a^*)) = 0,$$

so $a=0$. Similarly for $b\in B$.

DEFINITION 3.3. Using the form $\langle\ ,\ \rangle$ define the comultiplication Δ_B, counit ε_B, and antipode S_B as follows:

$$\begin{aligned} \Delta_B\colon B\to B\otimes B\colon &\qquad \langle a_1a_2, b\rangle = \langle a_1, b_{(1)}\rangle\langle a_2, b_{(2)}\rangle,\\ \varepsilon_B\colon B\to\mathbb{C}\colon &\qquad \varepsilon_B(b) = \langle 1, b\rangle = \lambda^{-1}\,d\tau(be_2),\\ S_B\colon B\to B\colon &\qquad \langle a, S_B(b)\rangle = \overline{\langle a^*, b^*\rangle}, \end{aligned}$$

for all $a, a_1, a_2\in A$ and $b\in B$. Similarly, we define Δ_A, ε_A, and S_A.

Clearly, $(B, \Delta_B, \varepsilon_B)$ (resp. $(A, \Delta_A, \varepsilon_A)$) becomes a coalgebra. Let us investigate the relations between the algebra and coalgebra structures on B.

4. WEAK KAC ALGEBRA STRUCTURE ON $M'\cap M_2$ (THE CASE OF A SCALAR WATATANI INDEX OF $\tau|_{M'\cap M_1}$)

LEMMA 4.1. *For all $a\in A$ and $b_1, b_2\in B$ we have*

$$\langle a, b_1b_2\rangle = \lambda^{-1}\langle E_{M_1}(b_2ae_2), b_1\rangle.$$

Proof. Using the definition of $\langle\ ,\ \rangle$ we have

$$\begin{aligned}\langle a, b_1b_2\rangle &= d\lambda^{-2}\tau(b_2ae_2e_1b_1)\\ &= d\lambda^{-3}\tau(E_{M_1}(b_2ae_2)\,e_2e_1b_1)\\ &= \lambda^{-1}\langle E_{M_1}(b_2ae_2), b_1\rangle.\end{aligned}$$

PROPOSITION 4.2. *Let* $\varepsilon_B^t(b)=\varepsilon_B(1_{(1)}b)\,1_{(2)}$. *Then* $\varepsilon_B^t(b)=\lambda^{-1}E_{M_1}(be_2)$ *and*

$$\langle a, \varepsilon_B^t(b)\rangle = d\lambda^{-2}\tau(ae_1be_2)=\lambda^{-1}\langle E_M(ae_1), b\rangle.$$

Proof. Using Lemma 4.1, definitions of Δ_B and ε_B, we have

$$\begin{aligned}\langle a, \varepsilon_B(1_{(1)}b)\,1_{(2)}\rangle &= \langle 1, 1_{(1)}b\rangle\langle a, 1_{(2)}\rangle\\ &= \langle \lambda^{-1}E_{M_1}(be_2), 1_{(1)}\rangle\langle a, 1_{(2)}\rangle\\ &= \langle \lambda^{-1}E_{M_1}(be_2)\,a, 1\rangle = \langle a, \lambda^{-1}E_{M_1}(be_2)\rangle,\end{aligned}$$

from where the first statement follows. For the second one, we have, using the λ-Markov property and the fact that e_2 commutes with M,

$$\begin{aligned}\langle a, \lambda^{-1}E_{M_1}(be_2)\rangle &= d\lambda^{-2}\tau(ae_1e_2\lambda^{-1}E_{M_1}(be_2))\\ &= d\lambda^{-2}\tau(ae_1\lambda^{-1}E_{M_1}(be_2)\,e_2)\\ &= d\lambda^{-2}\tau(ae_1be_2) = d\lambda^{-3}\tau(E_M(ae_1)\,e_1be_2)\\ &= d\lambda^{-3}\tau(E_M(ae_1)\,e_2e_1b) = \lambda^{-1}\langle E_M(ae_1), b\rangle.\end{aligned}$$

PROPOSITION 4.3. *For all* $b, c\in B$ *we have*

$$b_{(1)}\otimes\varepsilon_B^t(b_{(2)}) = 1_{(1)}b\otimes 1_{(2)}, \qquad b\varepsilon_B^t(c)=\varepsilon_B(b_{(1)}c)\,b_{(2)}.$$

Proof. For all $a_1, a_2\in A$ we compute, using Lemma 4.1 and Proposition 4.2:

$$\begin{aligned}\langle a_1, b_{(1)}\rangle\langle a_2, \varepsilon_B^t(b_{(2)})\rangle &= \langle a_1\lambda^{-1}E_M(a_2e_1), b\rangle\\ &= \lambda^{-2}\langle E_{M_1}(ba_1E_M(a_2e_1)\,e_2), 1\rangle\\ &= \lambda^{-2}\langle E_{M_1}(ba_1e_2)\,E_M(a_2e_1), 1\rangle\\ &= d\lambda^{-3}\tau(E_{M_1}(ba_1e_2)\,E_M(a_2e_1)\,e_1)\\ &= d\lambda^{-2}\tau(E_{M_1}(ba_1e_2)\,a_2e_1)\\ &= \lambda^{-1}\langle E_{M_1}(ba_1e_2)\,a_2, 1\rangle\\ &= \langle \lambda^{-1}E_{M_1}(ba_1e_2), 1_{(1)}\rangle\langle a_2, 1_{(2)}\rangle\\ &= \langle a_1, 1_{(1)}b\rangle\langle a_2, 1_{(2)}\rangle,\end{aligned}$$

$$\begin{aligned}\langle a, b\varepsilon^t_B(c)\rangle &= \langle \varepsilon^t_B(c)\, a, b\rangle \\ &= \langle \lambda^{-1} E_{M_1}(ce_2)\, a, b\rangle \\ &= \langle \lambda^{-1} E_{M_1}(ce_2), b_{(1)}\rangle \langle a, b_{(2)}\rangle \\ &= \langle 1, b_{(1)}c\rangle \langle a, b_{(2)}\rangle \\ &= \langle a, \varepsilon_B(b_{(1)}c)\, b_{(2)}\rangle.\end{aligned}$$

Since the duality is non-degenerate, the result follows.

The antipode map assigns to each $b \in B$ a unique element $S_B(b) \in B$ such that $\tau(ae_2e_1S_B(b)) = \tau(be_1e_2a)$ for all $a \in A$, or, equivalently,

$$E_{M_1}(be_1e_2) = E_{M_1}(e_2e_1S_B(b)).$$

Taking $a = e_1$ and using the λ-Markov property of e_1 we get $\tau \circ S_B = \tau$. Similarly, $E_{M'}(S_A(a)\, e_2e_1) = E_{M'}(e_1e_2a)$ and $\tau \circ S_A = \tau$.

Remark 4.4. Note that the condition $E_{M_1}(be_1e_2) = E_{M_1}(e_2e_1S_B(b))$ implies that

$$E_{M_1}(bxe_2) = E_{M_1}(e_2xS_B(b)) \qquad \text{for all} \quad x \in M_1.$$

Indeed, any $x \in M_1$ can be written as $x = \sum x_ie_1y_i$ with $x_i, y_i \in M \subset B'$. Similarly, we have

$$E_{M'}(S_A(a)\, ye_1) = E_{M'}(e_1ya) \qquad \text{for all} \quad y \in M'.$$

PROPOSITION 4.5. *The following identities hold*:

(i) $S_B(b) = \lambda^{-3}E_{M'}(e_1e_2E_{M_1}(be_1e_2))$,

(ii) $S_B(B_s) = B_t$,

(iii) $S^2_B(b) = b$ *and* $S_B(b)^* = S_B(b^*)$,

(iv) $S_B(bc) = S_B(c)\, S_B(b)$ *and* $\Delta_B(S_B(b)) = \varsigma(S_B \otimes S_B)\, \Delta_B(b)$.

Proof. (i) We have

$$\begin{aligned}S_B(b) &= \lambda^{-1}E_{M'}(e_1S_B(b)) \\ &= \lambda^{-2}E_{M'}(e_1e_2e_1S_B(b)) \\ &= \lambda^{-3}E_{M'}(e_1e_2E_{M_1}(e_2e_1S_B(b))) \\ &= \lambda^{-3}E_{M'}(e_1e_2E_{M_1}(be_1e_2)).\end{aligned}$$

(ii) If $z \in B_s$ then $ze_2 = e_2 z$ and by the explicit formula (i) we get,

$$\begin{aligned} S_B(z) &= \lambda^{-3} E_{M'}(e_1 e_2 E_{M_1}(e_1 z e_2)) \\ &= \lambda^{-2} E_{M'}(e_1 E_{M_1}(z e_2)) \\ &= \lambda^{-1} E_{M_1}(z e_2) \\ &= \varepsilon_B^t(z) \in B_t. \end{aligned}$$

(iii) Since E_{M_1} preserves $*$, we get $E_{M_1}(e_2 e_1 b^*) = E_{M_1}(S_B(b)^* \, e_2 e_1)$, from where $S_B(S_B(b)^*)^* = b$. Next, using Lemma 4.1, Remark 4.4, and the λ-Markov property of e_2, we compute

$$\begin{aligned} \tau(a e_2 e_1 b) &= \lambda^{-1} \tau(E_{M_1}(b a e_2) \, e_2 e_1) \\ &= \lambda^{-1} \tau(E_{M_1}(e_2 a S_B(b)) \, e_2 e_1) \\ &= \lambda^{-1} \tau(e_2 E_{M_1}(e_2 a S_B(b)) \, e_1) \\ &= \tau(e_2 a S_B(b) \, e_1) \\ &= \tau(S_B(b) \, e_1 e_2 a) \\ &= \tau(a e_2 e_1 S_B^2(b)). \end{aligned}$$

therefore, $S_B^2(b) = b$ and $S_B(b)^* = S_B(b^*)$.

(iv) Using Remark 4.4, we have

$$\begin{aligned} \tau(a e_2 e_1 S_B(bc)) &= \tau(b c e_1 e_2 a) \\ &= \lambda^{-1} \tau(c e_1 e_2 E_{M_1}(e_2 a b)) \\ &= \lambda^{-1} \tau(E_{M_1}(e_2 a b) \, e_2 e_1 S_B(c)) \\ &= \lambda^{-1} \tau(E_{M_1}(S_B(b) \, a e_2) \, e_2 e_1 S_B(c)) \\ &= \tau(a e_2 e_1 S_B(c) \, S_B(b)), \end{aligned}$$

which proves that $\langle a, S_B(bc) \rangle = \langle a, S_B(c) \, S_B(b) \rangle$. Similarly, one can prove that S_A is anti-multiplicative, and since $\langle a, S_B(b) \rangle = \langle S_A(a), b \rangle$, the second part of (iv) follows.

Let $\{f_{kl}^\alpha\}$ be a system of matrix units in $B_t = M' \cap M_1 = \bigoplus_\alpha M_{m_\alpha}(\mathbb{C})$, where $\sum m_\alpha^2 = d$, and let $\tau_\alpha = \tau(f_{kk}^\alpha)$.

PROPOSITION 4.6. *The explicit formula for $\Delta_B(1)$ is*

$$\Delta_B(1)=\sum_{\alpha kl}\frac{1}{d\tau_\alpha}S_B(f^\alpha_{kl})\otimes f^\alpha_{lk}.$$

In particular, $\Delta_B(1)$ is a positive element in $B_s\otimes B_t$.

Proof. Note that the map $x\mapsto\sum_{\alpha kl}(\tau(xf^\alpha_{lk})/\tau_\alpha)\,f^\alpha_{kl}$ defines the τ-preserving conditional expectation on B_t. For all $a_1, a_2\in A$ we have

$$\begin{aligned}
&\sum_{\alpha kl}\frac{1}{d\tau_\alpha}\langle a_1, S_B(f^\alpha_{kl})\rangle\langle a_2, f^\alpha_{lk}\rangle\\
&\quad=d^2\lambda^{-4}\sum_{\alpha kl}\frac{1}{d\tau_\alpha}\tau(a_1e_2e_1S_B(f^\alpha_{kl}))\,\tau(a_2e_2e_1f^\alpha_{lk})\\
&\quad=d\lambda^{-3}\sum_{\alpha kl}\tau(f^\alpha_{kl}e_1e_2a_1)\frac{\tau(a_2e_1f^\alpha_{lk})}{\tau_\alpha}\\
&\quad=d\lambda^{-3}\tau(E_{M'}(a_2e_1)\,e_1e_2a_1)\\
&\quad=d\lambda^{-2}\tau(a_1a_2e_1e_2)=\langle a_1a_2, 1\rangle,
\end{aligned}$$

which proves the statement.

COROLLARY 4.7. $\Delta_B(1)=\sum_{\alpha kl}(1/m_\alpha)\,S_B(f^\alpha_{kl})\otimes f^\alpha_{lk}H$, *where H is canonically defined by*

$$H=S_B(1_{(1)})\,1_{(2)}=\frac{1}{d}\sum_\alpha\frac{m_\alpha}{\tau_\alpha}\sum_k f^\alpha_{kk}=\frac{1}{d}\,\mathit{Index}\,\tau|_{M'\cap M_1}\in Z(B_t),$$

where $\mathit{Index}\,\tau|_{M'\cap M_1}$ is the Watatani index [26] of the restriction of τ to $M'\cap M_1$ and $Z(\cdot)$ denotes the center of the algebra. We also have $\tau(H)=1$.

PROPOSITION 4.8. *For all $b\in B$ we have $\varepsilon^t_B(b_{(1)})\,b_{(2)}=Hb$.*

Proof. Applying $E_{M'}$ to both sides of $E_{M_1}(b^*e_1e_2)=E_{M_1}(e_2e_1S_B(b^*))$ and using the relation $E_{M_1}\circ E_{M'}=E_{M'}\circ E_{M_1}$, we get

$$E_{M_1}(b^*e_2)=E_{M_1}(e_2S_B(b^*))$$

which means that $\varepsilon^t_B(b^*)=\varepsilon^t_B(S_B(b))^*$. Using Propositions 4.3, 4.5(iv), and Corollary 4.7 we get $S_B(b_{(1)})\,\varepsilon^t_B(b_{(2)})=S_B(b)\,H$, from where $HS_B(b^*)=\varepsilon^t_B(b_{(2)})^*\,S(b^*_{(1)})$. Replacing $S_B(b^*)$ by b, we get the result.

Let $\{s^{\alpha}_{jk}\}$ be a basis consisting of matrix units of A and $\{v^{\alpha}_{jk}\}$ be a basis of comatrix units of B dual to each other, i.e.,

$$\langle v^{\alpha}_{jk}, s^{\beta}_{pq}\rangle = \delta_{\alpha\beta}\, \delta_{jp}\, \delta_{kq}.$$

We have $\Delta_B(v^{\alpha}_{jk}) = \sum_l v^{\alpha}_{jl} \otimes v^{\alpha}_{lk}$ and $\varepsilon_B(v^{\alpha}_{jk}) = \delta_{jk}$.

LEMMA 4.9. *Let $/\alpha/ = \tau(s^{\alpha}_{kk})$. The following identities hold true:*

(i) $E_{M_1}(e_2 e_1 v^{\alpha}_{jk}) = d^{-1}\lambda^2\, /\alpha/^{-1}\, s^{\alpha}_{kj}$,
(ii) $E_{M_1}(v^{\alpha}_{jk} e_1 e_2) = d^{-1}\lambda^2\, /\alpha/^{-1}\, S_A(s^{\alpha}_{kj})$,
(iii) $\lambda^{-1} E_{M'}(S_A(s^{\beta}_{pq})\, v^{\alpha}_{ij} e_1) = \delta_{\alpha\beta}\, \delta_{ip} v^{\alpha}_{qj}$,
(iv) $S_B(v^{\alpha}_{jk}) = (v^{\alpha}_{kj})^*$.

Proof. (i) We can directly compute:

$$\begin{aligned} d\lambda^{-2}\, /\alpha/\, \tau(s^{\beta}_{pq} E_{M_1}(e_2 e_1 v^{\alpha}_{jk})) &= /\alpha/\, \langle s^{\beta}_{pq}, v^{\alpha}_{jk}\rangle \\ &= /\alpha/\, \delta_{\alpha\beta}\, \delta_{jp}\, \delta_{kq} \\ &= \tau(s^{\alpha}_{kj} s^{\beta}_{pq}), \end{aligned}$$

therefore, we have $E_{M_1}(e_2 e_1 v^{\alpha}_{jk}) = d^{-1}\lambda^2\, /\alpha/^{-1}\, s^{\alpha}_{kj}$ by the faithfulness of τ.

(ii) Similarly to (i), we compute

$$\begin{aligned} d\lambda^{-2}\, /\alpha/\, \tau(E_{M_1}(v^{\alpha}_{jk} e_1 e_2)\, S_A(s^{\beta}_{pq})) &= /\alpha/\, \langle s^{\beta}_{pq}, v^{\alpha}_{jk}\rangle \\ &= /\alpha/\, \delta_{\alpha\beta}\, \delta_{jp}\, \delta_{kq} \\ &= \tau(S_A(s^{\alpha}_{kj})\, S_A(s^{\beta}_{pq})), \end{aligned}$$

and since S_A is injective, the result follows.

(iii) Using Remark 4.4, we have

$$\begin{aligned} \langle s^{\gamma}_{rt}, \lambda^{-1} E_{M'}(S_A(s^{\beta}_{pq})\, v^{\alpha}_{ij} e_1)\rangle &= \langle s^{\gamma}_{rt}, \lambda^{-1} E_{M'}(e_1 v^{\alpha}_{ij} s^{\beta}_{pq})\rangle \\ &= \langle s^{\beta}_{pq} s^{\gamma}_{rt}, v^{\alpha}_{ij}\rangle \\ &= \delta_{\alpha\gamma}\, \delta_{qr}\, \delta_{ip}\, \delta_{\alpha\beta}\, \delta_{tj} \\ &= \delta_{\alpha\beta}\, \delta_{ip} \langle s^{\gamma}_{rt}, v^{\alpha}_{qj}\rangle. \end{aligned}$$

(iv) Using part (i), we have

$$\begin{aligned} E_{M_1}((v^{\alpha}_{kj})^*\, e_1 e_2) = E_{M_1}(e_2 e_1 v^{\alpha}_{kj})^* &= d^{-1}\lambda^2\, /\alpha/^{-1}\, s^{\alpha}_{kj} \\ &= E_{M_1}(e_2 e_1 v^{\alpha}_{jk}) = E_{M_1}(S_B(v^{\alpha}_{jk})\, e_1 e_2), \end{aligned}$$

and the result follows from the injectivity of the map $b \mapsto E_{M_1}(b e_1 e_2)$.

COROLLARY 4.10. *$\Delta_B(b^*) = \Delta_B(b)^{*\otimes *}$, i.e., Δ_B is a $*$-preserving map.*

Proof. Using Lemmas 4.9(iv) and 4.5(iv), we have

$$\begin{aligned}\Delta_B((v^\alpha_{jk})^*) &= \Delta_B(S_B(v^\alpha_{kj})) = \Sigma_i S_B(v^\alpha_{ij}) \otimes S_B(v^\alpha_{ki})\\ &= \Sigma_i (v^\alpha_{ji})^* \otimes (v^\alpha_{ik})^* = \Delta_B(v^\alpha_{jk})^{*\otimes *}.\end{aligned}$$

PROPOSITION 4.11. *$v^\alpha_{ij} e_1 = \lambda^{-1} \sum_k E_{M_1}(v^\alpha_{ik} e_1 e_2)\, H^{-1} v^\alpha_{kj}$.*

Proof. By Lemma 4.9(ii), all we need to show is

$$v^\alpha_{ij} e_1 = d^{-1} \lambda\, /\alpha/^{-1} \sum_k S_A(s^\alpha_{ki})\, H^{-1} v^\alpha_{kj}.$$

Since $N' \cap M_2$ is spanned by the elements of the form $v^\gamma_{rt} S_A(s^\beta_{pq})$, it suffices to verify that

$$\tau(v^\gamma_{rt} S_A(s^\beta_{pq})\, v^\alpha_{ij} e_1) = d^{-1} \lambda\, /\alpha/^{-1} \sum_k \tau(v^\gamma_{rt} S_A(s^\beta_{pq})\, S_A(s^\alpha_{ki})\, H^{-1} v^\alpha_{kj}),$$

or, equivalently,

$$E_{M'}(S_A(s^\beta_{pq})\, v^\alpha_{ij} e_1) = \delta_{\alpha\beta}\, \delta_{ip}\, \lambda d^{-1}\, /\alpha/^{-1} \sum_k E_{M'}(S_A(s^\beta_{kq})\, H^{-1} v^\alpha_{kj}).$$

Using Lemma 4.9(iii), we can reduce the proof to the verification of the relation

$$v^\alpha_{qj} = d^{-1}\, /\alpha/^{-1} \sum_k E_{M'}(S_A(s^\alpha_{kq}))\, H^{-1} v^\alpha_{kj}.$$

By Lemma 4.9(ii),

$$E_{M'}(S_A(s^\alpha_{kq})) = d\lambda^{-2}\, /\alpha/\, E_{M'} \circ E_{M_1}(v^\alpha_{qk} e_1 e_2) = d\lambda^{-1}\, /\alpha/\, E_{M_1}(v^\alpha_{qk} e_2),$$

therefore the previous relation is equivalent to

$$v^\alpha_{qj} = \lambda^{-1} \sum_k E_{M_1}(v^\alpha_{qk} e_2)\, H^{-1} v^\alpha_{kj}.$$

Since $H \in Z(B_t)$, this is precisely Proposition 4.8 with $b = v^\alpha_{qj}$, so the proof is complete.

COROLLARY 4.12. *$bx=\lambda^{-1}E_{M_1}(b_{(1)}xe_2)\,H^{-1}b_{(2)}$ for all $b\in B$ and $x\in M_1$.*

Proof. Proposition 4.11 implies that $be_1=\lambda^{-1}E_{M_1}(b_{(1)}e_1e_2)\,H^{-1}b_{(2)}$ for all $b\in B$. As in Remark 4.4, any $x\in M_1$ can be written as a finite sum $x=\sum x_ie_2y_i$ with $x_i,\ y_i\in M\subset B'$, therefore, we have

$$\begin{aligned}
bx=\sum x_ibe_1y_i&=\sum x_i\lambda^{-1}E_{M_1}(b_{(1)}e_1e_2)\,H^{-1}b_{(2)}y_i\\
&=\lambda^{-1}E_{M_1}\left(b_{(1)}\sum x_ie_1y_ie_2\right)H^{-1}b_{(2)}\\
&=\lambda^{-1}E_{M_1}(b_{(1)}xe_2)\,H^{-1}b_{(2)}.
\end{aligned}$$

PROPOSITION 4.13. *For all $x,\ y\in M_1$,*

$$E_{M_1}(bxye_2)=\lambda^{-1}E_{M_1}(b_{(1)}xe_2)\,H^{-1}E_{M_1}(b_{(2)}ye_2).$$

Proof. Multiplying the formula from Corollary 4.12 on the right by ye_2 and taking E_{M_1} from both sides we get the required identity.

PROPOSITION 4.14. *$\Delta_B(bc)=\Delta_B(b)(1\otimes H^{-1})\,\Delta_B(c)$, for all $b,\ c\in B$.*

Proof. By Lemma 4.1 and Proposition 4.13 we have for all $a_1, a_2\in A$:

$$\begin{aligned}
\langle a_1a_2, bc\rangle&=\langle\lambda^{-1}E_{M_1}(ca_1a_2e_2), b\rangle\\
&=\langle\lambda^{-2}E_{M_1}(c_{(1)}a_1e_2)\,H^{-1}E_{M_1}(c_{(2)}a_2e_2), b\rangle\\
&=\langle\lambda^{-1}E_{M_1}(c_{(1)}a_1e_2), b_{(1)}\rangle\langle\lambda^{-1}E_{M_1}(H^{-1}c_{(2)}a_2e_2), b_{(2)}\rangle\\
&=\langle a_1, b_{(1)}c_{(1)}\rangle\langle a_2, b_{(2)}H^{-1}c_{(2)}\rangle,
\end{aligned}$$

from where $\Delta_B(bc)=b_{(1)}c_{(1)}\otimes b_{(2)}H^{-1}c_{(2)}$ which is the result.

PROPOSITION 4.15. *$b_{(1)}S_B(b_{(2)}H^{-1})=\varepsilon_B^t(b)$.*

Proof. Using Corollary 4.10, Proposition 4.13, and Proposition 4.2 we have

$$\begin{aligned}
\langle a, b_{(1)}S_B(b_{(2)}H^{-1})\rangle&=d\lambda^{-3}\tau(E_{M_1}(S_B(b_{(2)}H^{-1})\,ae_2)\,e_2e_1b_{(1)})\\
&=d\lambda^{-3}\tau(E_{M_1}(e_2ab_{(2)}H^{-1})\,e_2e_1b_{(1)})\\
&=d\lambda^{-3}\tau(E_{M_1}(e_2ab_{(2)}H^{-1})\,E_{M_1}(e_2e_1b_{(1)}))\\
&=d\lambda^{-2}\tau(E_{M_1}(e_2ae_1b))\\
&=\langle a, \varepsilon_B^t(b)\rangle.
\end{aligned}$$

The next Corollary summarizes the properties of Δ_B, ε_B, and S_B.

COROLLARY 4.16. *$(\Delta_B, \varepsilon_B)$ defines a coalgebra structure on B such that*

$$\Delta_B(bc) = \Delta_B(b)(1 \otimes H^{-1})\, \Delta_B(c) \qquad \Delta_B(b^*) = \Delta_B(b)^{*\otimes *},$$

the map ε_B^t, defined by $\varepsilon_B^t(b) = \varepsilon_B(1_{(1)}b)\, 1_{(2)}$, satisfies the relations

$$b_{(1)} \otimes \varepsilon_B^t(b_{(2)}) = 1_{(1)}b \otimes 1_{(2)}, \qquad b\varepsilon_B^t(c) = \varepsilon_B(b_{(1)}c)\, b_{(2)},$$

and S_B is a $*$-preserving anti-algebra and anti-coalgebra involution such that

$$b_{(1)} S_B(b_{(2)} H^{-1}) = \varepsilon_B^t(b),$$

for all $b, c \in B$.

THEOREM 4.17. *The following conditions are equivalent*:

(i) *$(B, \Delta_B, \varepsilon_B, S_B)$ is a weak Kac algebra with the Haar projection e_2 and the normalized Haar trace $\phi(b) = d\tau(b)$, $b \in B$,*

(ii) *$H = 1$.*

Moreover, if these conditions are satisfied, then λ^{-1} is an integer.

Proof. (i) $\Rightarrow$ (ii). If Δ_B is an algebra homomorphism, then we must have $\Delta_B(1) = \Delta_B(1)(1 \otimes H^{-1})$, and applying $(\varepsilon_B \otimes \mathrm{id})$ we get $H^{-1} = 1$.

(ii) $\Rightarrow$ (i). Clearly, if $H = 1$, then $(B, \Delta_B, \varepsilon_B, S_B)$ is a weak Kac algebra. For all $b \in B$ we have, by Proposition 4.2:

$$be_2 = \lambda^{-1} E_{M_1}(be_2)\, e_2 = \varepsilon_B^t(b)\, e_2,$$

and we easily get $S_B(e_2) = e_2$ and $\varepsilon_B^t(e_2) = 1$, so e_2 is the Haar projection in B.

Next, since $\tau(b) = d^{-1}\langle e_1, b\rangle$, we have by Proposition 4.2:

$$\begin{aligned}
\langle a, \varepsilon_B^t(b_{(1)})\, \tau(b_{(2)})\rangle &= d^{-1}\langle a, \varepsilon_B^t(b_{(1)})\rangle \langle e_1, b_{(2)}\rangle \\
&= d^{-1}\langle \lambda^{-1} E_M(ae_1), b_{(1)}\rangle \langle e_1, b_{(2)}\rangle \\
&= d^{-1}\langle \lambda^{-1} E_M(ae_1)\, e_1, b\rangle \\
&= d^{-1}\langle ae_1, b\rangle = \langle a, b_{(1)}\tau(b_{(2)})\rangle,
\end{aligned}$$

from where we get $\varepsilon_B^t(b_{(1)})\, \phi(b_{(2)}) = b_{(1)}\phi(b_{(2)})$. Also, $\tau(S_B(b)) = \tau(b)$ and $\tau \circ \varepsilon_B^t(b) = \lambda^{-1}\tau(E_{M_1}(be_2)) = d^{-1}\varepsilon_B(b)$, therefore $\phi \circ S_B = \phi$ and $\phi \circ \varepsilon_B^t = \varepsilon_B$. Thus, ϕ is the normalized Haar trace.

If $H=1$, then the "trace vector" of the restriction of τ on B_t is given by $\boldsymbol{\tau}=(1/d)(m_1, m_2, ...)$, so the components of $\boldsymbol{\tau}$ are rational numbers. Let Λ be the inclusion matrix of $B_t \subset B$, then

$$\Lambda\Lambda^t\boldsymbol{\tau}=\lambda^{-1}\boldsymbol{\tau}.$$

Since all entries of $\Lambda\Lambda^t$ and $\boldsymbol{\tau}$ are rational, λ^{-1} must be rational. On the other hand, λ^{-1} is an algebraic integer as an eigenvalue of the integer matrix $\Lambda\Lambda^t$. Therefore, λ^{-1} is integer.

Proposition 4.18. *If $N \subset M$ is a depth 2 inclusion of II_1 factors such that $[M:N]$ is a square free integer (i.e., $[M:N]$ is an integer which has no divisors of the form n^2, $n>1$), then $N' \cap M=\mathbb{C}$, and there is a (canonical) minimal action of a Kac algebra B on M_1 such that $M_2 \cong M_1 \rtimes B$ and $M=M_1^B$.*

Proof. It suffices to show that $N \subset M$ is irreducible, since the rest follows from [23]. Let q be a minimal projection in $M' \cap M_1$, then the reduced inclusion $qM \subset qM_1q$ is of finite depth [1]. Since any finite depth inclusion is extremal (see, e.g., [21], 1.3.6) we have

$$[qM_1q : qM]=\tau(q)^2\,[M_1 : M]=\tau(q)^2\,[M : N],$$

by ([19], Corollary 4.5).

We claim that $\tau(q)$ is a rational number. Indeed, it is well-known that the Perron–Frobenius eigenspace of the non-negative matrix $\Lambda\Lambda^t$ is 1-dimensional [8]. Letting one of the components of a corresponding eigenvector τ to be equal to 1, one can recover the rest of components from the system of linear equations with integer coefficients. Thus, we have that all components of $\boldsymbol{\tau}$ are rational; clearly, the normalization condition $\tau(1)=1$ does not change this property.

Therefore, the index $[qM_1q : qM]$ is a rational number. On the other hand, it must be an algebraic integer, since the depth is finite. Therefore, $[qM_1q : qM]$ is an integer. Since $[M:N]$ is square free, we must have $\tau(q)=1$, which means that $M' \cap M_1$ and $N' \cap M$ are 1-dimensional.

Corollary 4.19. *If $N \subset M$ is a depth 2 inclusion of II_1 factors such that $[M:N]=p$ is prime, then $N' \cap M=\mathbb{C}$, and there is an outer action of the cyclic group $G=\mathbb{Z}/p\mathbb{Z}$ on M_1 such that $M_2 \cong M_1 \rtimes G$ and $M=M_1^G$.*

Proof. By Proposition 4.18, B must be a Kac algebra of prime dimension p. But it is known that any such an algebra is a group algebra of the cyclic group $G=\mathbb{Z}/p\mathbb{Z}$ [12].

5. WEAK C^*-HOPF ALGEBRA STRUCTURE ON $M' \cap M_2$ (THE GENERAL CASE)

When $H \neq 1$, $(B, \Delta_B, \varepsilon_B, S_B)$ is no longer a weak Kac algebra (for instance, Δ_B is not a homomorphism). However, it is possible to deform the structure maps in such a way that B becomes a weak C^*-Hopf algebra.

DEFINITION 5.1. Let us define the following operations on B:

$$\begin{array}{rll} \textit{involution} & \dagger: B \to B: & b^\dagger = S_B(H)^{-1}\, b^* S_B(H), \\ \textit{comultiplication} & \tilde{\Delta}: B \to B \otimes B: & \tilde{\Delta}(b) = (1 \otimes H^{-1})\, \Delta_B(b), \qquad \textit{i.e.}, \\ & & b_{(\tilde{1})} \otimes b_{(\tilde{2})} = b_{(1)} \otimes H^{-1} b_{(2)} \\ \textit{counit} & \tilde{\varepsilon}: B \to \mathbb{C}: & \tilde{\varepsilon}(b) = \varepsilon_B(Hb), \\ \textit{antipode} & \tilde{S}: B \to B: & \tilde{S}(b) = S_B(HbH^{-1}). \end{array}$$

Clearly, $\dagger$ defines a C^*-algebra structure on B (we will still denote this new C^*-algebra by B). Our goal is to show that $(B, \tilde{\Delta}, \tilde{\varepsilon}, \tilde{S})$ is a weak C^*-Hopf algebra. The proof of this fact consists of a verification of all the axioms from Section 2. We will need the following technical lemma.

LEMMA 5.2. *For all $b \in B$ and $z \in B_t$ we have*

(i) $\varepsilon_B^t(zb) = z\varepsilon_B^t(b)$,

(ii) $b_{(1)} z \otimes b_{(2)} = (bz)_{(1)} \otimes (bz)_{(2)}$,

(iii) $b_{(1)} S_B(z) \otimes b_{(2)} = b_{(1)} \otimes b_{(2)} z$,

Proof. Part (i) is clear from Proposition 4.2. Next, recall that $B_t = A \cap B$, and compute

$$\langle a_1, b_{(1)} z \rangle \langle a_2, b_{(2)} \rangle = \langle z a_1 a_2, b \rangle = \langle a_1 a_2, bz \rangle, \qquad a_1, a_2 \in A,$$

which gives (ii). Finally, using the properties of S_B we have

$$\begin{aligned} \langle a_1, b_{(1)} S_B(z) \rangle \langle a_2, b_{(2)} \rangle &= \langle a_1, S_B(z S_B(b_{(1)})) \rangle \langle a_2, b_{(2)} \rangle \\ &= \overline{\langle a_1^*, S_B(b_{(1)}^*)\, z^* \rangle} \langle a_2, b_{(2)} \rangle \\ &= \overline{\langle (a_1 z)^*, S_B(b_{(1)}^*) \rangle} \langle a_2, b_{(2)} \rangle \\ &= \langle a_1 z, b_{(1)} \rangle \langle a_2, b_{(2)} \rangle \\ &= \langle a_1 z a_2, b \rangle \\ &= \langle a_1, b_{(1)} \rangle \langle a_2, b_{(2)} z \rangle, \end{aligned}$$

from where (iii) follows.

PROPOSITION 5.3. *$(B, \tilde{\Delta}, \tilde{\varepsilon})$ is a coalgebra.*

Proof. Let us check the coassociativity of $\tilde{\Delta}$. Using Lemma 5.2 and the fact that $H \in B_t$ we compute for all $b \in B$:

$$\begin{aligned}
(\tilde{\Delta} \otimes \mathrm{id})\, \tilde{\Delta}(b) &= \tilde{\Delta}(b_{(1)}) \otimes H^{-1} b_{(2)} \\
&= b_{(1)} \otimes H^{-1} b_{(2)} \otimes H^{-1} b_{(3)} \\
&= b_{(1)} \otimes (H^{-1} b_{(2)})_{(1)} \otimes H^{-1} (H^{-1} b_{(2)})_{(2)} \\
&= b_{(1)} \otimes \tilde{\Delta}(H^{-1} b_{(2)}) \\
&= (\mathrm{id} \otimes \tilde{\Delta})\, \tilde{\Delta}(b).
\end{aligned}$$

Next, we check the counit axioms:

$$\begin{aligned}
(\tilde{\varepsilon} \otimes \mathrm{id})\, \tilde{\Delta}(b) &= \varepsilon(H b_{(1)})\, H^{-1} b_{(2)} = \varepsilon((Hb)_{(1)})\, H^{-1} (Hb)_{(2)} = b, \\
(\mathrm{id} \otimes \tilde{\varepsilon})\, \tilde{\Delta}(b) &= b_{(1)} \varepsilon(H H^{-1} b_{(2)}) = b.
\end{aligned}$$

PROPOSITION 5.4. *$\tilde{\Delta}$ is a $\dagger$-homomorphism.*

Proof. Using the properties of Δ_B from Corollary 4.16 and Lemma 5.2 we have:

$$\begin{aligned}
\tilde{\Delta}(bc) &= (1 \otimes H^{-1})\, \Delta_B(bc) \\
&= (1 \otimes H^{-1})\, \Delta_B(b)(1 \otimes H^{-1})\, \Delta_B(c) = \tilde{\Delta}(b)\, \tilde{\Delta}(c), \\
\tilde{\Delta}(b^{\dagger}) &= \tilde{\Delta}(S_B(H)^{-1}\, b^* S_B(H)) \\
&= (S_B(H)^{-1}\, b^* S_B(H))_{(1)} \otimes H^{-1} (S_B(H)^{-1}\, b^* S_B(H))_{(2)} \\
&= S_B(H)^{-1}\, b^*_{(1)} \otimes S_B(H)^{-1}\, b^*_{(2)} S_B(H) = (S_B(H)^{-1}\, b_{(1)})^{\dagger} \otimes b^{\dagger}_{(2)} \\
&= b^{\dagger}_{(1)} \otimes (H^{-1} b_{(2)})^{\dagger} = \tilde{\Delta}(b)^{\dagger \otimes \dagger}.
\end{aligned}$$

PROPOSITION 5.5. *Let $\tilde{\varepsilon}^t(b) = \tilde{\varepsilon}(1_{(\tilde{1})} b)\, 1_{(\tilde{2})}$. Then for all $b, c \in B$:*

$$b \tilde{\varepsilon}^t(c) = \tilde{\varepsilon}(b_{(\tilde{1})} c)\, b_{(\tilde{2})}, \qquad b_{(\tilde{1})} \otimes \tilde{\varepsilon}^t(b_{(\tilde{2})}) = 1_{(\tilde{1})} b \otimes 1_{(\tilde{2})},$$

Proof. First, we compute, using Lemma 5.2 and Proposition 4.3:

$$\tilde{\varepsilon}^t(b) = \varepsilon(H 1_{(1)} b)\, H^{-1} 1_{(2)} = \varepsilon(H_{(1)} b)\, H_{(2)} H^{-1} = H \varepsilon^t_B(b)\, H^{-1} = \varepsilon^t_B(b).$$

Using this relation, Lemma 5.2, and properties of ε_B^t from Corollary 4.16 we have

$$\begin{aligned}
b_{(\tilde{1})} \otimes \tilde{\varepsilon}^t(b_{(\tilde{2})}) &= b_{(1)} \otimes \varepsilon_B^t(H^{-1} b_{(2)}) = b_{(1)} \otimes H^{-1} \varepsilon_B^t(b_{(2)}) \\
&= 1_{(1)} b \otimes H^{-1} 1_{(2)} = 1_{(\tilde{1})} b \otimes 1_{(\tilde{2})}, \\
b\tilde{\varepsilon}^t(c) &= b\varepsilon_B^t(c) = H^{-1}(Hb)\, \varepsilon_B^t(c) \\
&= H^{-1} \varepsilon_B((Hb)_{(1)}\, c)(Hb)_{(2)} = \varepsilon_B(Hb_{(1)} c)\, H^{-1} b_{(2)} \\
&= \tilde{\varepsilon}(b_{(\tilde{1})} c)\, b_{(\tilde{2})}.
\end{aligned}$$

PROPOSITION 5.6. *$\tilde{S}$ is a linear anti-multiplicative and anti-comultiplicative map such that*

$$b_{(\tilde{1})} \tilde{S}(b_{(\tilde{2})}) = \tilde{\varepsilon}^t(b).$$

Moreover, $(\tilde{S} \circ \dagger)^2 = \mathrm{id}$ *and* $\tilde{S}^2(b) = GbG^{-1}$, *where* $G = \tilde{S}(H)^{-1}\, H$.

Proof. Using Corollary 4.16, Lemma 5.2 and definitions of $\tilde{S}$ and $\dagger$, we have:

$$\begin{aligned}
\tilde{S}(bc) &= S_B(HbcH^{-1}) = S_B(HcH^{-1})\, S_B(HbH^{-1}) = \tilde{S}(c)\, \tilde{S}(b), \\
\tilde{S}(b)_{(\tilde{2})} \otimes \tilde{S}(b)_{(\tilde{1})} &= H^{-1} S_B(HbH^{-1})_{(2)} \otimes S_B(HbH^{-1})_{(1)} \\
&= S_B(HbH^{-1})_{(2)} \otimes S_B(H^{-1})\, S_B(HbH^{-1})_{(1)} \\
&= S_B((HbH^{-1})_{(1)}) \otimes S_B(H^{-1})\, S_B((HbH^{-1})_{(2)}) \\
&= S_B(Hb_{(1)} H^{-1}) \otimes S_B(b_{(2)} H^{-1}) \\
&= \tilde{S}(b_{(1)}) \otimes \tilde{S}(H^{-1} b_{(2)}) \\
&= \tilde{S}(b_{(\tilde{1})}) \otimes \tilde{S}(b_{(\tilde{2})}), \\
b_{(\tilde{1})} \tilde{S}(b_{(\tilde{2})}) &= b_{(1)} S_B(b_{(2)} H^{-1}) = \varepsilon_B^t(b) = \tilde{\varepsilon}^t(b),
\end{aligned}$$

from where the first part of Proposition follows. Next, we can compute

$$\begin{aligned}
\tilde{S}(b^\dagger) &= S_B(Hb^\dagger H^{-1}) = S_B(HS_B(H)^{-1}\, b^* S_B(H)\, H^{-1}) \\
&= S_B(H)^{-1}\, HS_B(b^*)\, H^{-1} S_B(H), \\
\tilde{S}(\tilde{S}(b^\dagger)^\dagger) &= \tilde{S}((S_B(H)^{-1}\, HS_B(b^*)\, H^{-1} S_B(H))^\dagger) = \tilde{S}(H^{-1} S_B(b)\, H) \\
&= S_B(S_B(b)) = b,
\end{aligned}$$

therefore $(\tilde{S}\circ\dagger)^2=\mathrm{id}$. Finally, since $S_B(H)=\tilde{S}(H)$, we get

$$\tilde{S}^2(b)=\tilde{S}(S_B(HbH^{-1}))=S_B(HS_B(HbH^{-1})\,H^{-1})$$
$$=S_B(H)^{-1}\,HbH^{-1}S_B(H)=GbG^{-1}.$$

Thus, we can state the main result of this section.

THEOREM 5.7. *$(B,\tilde{\Delta},\tilde{\varepsilon},\tilde{S})$ is a weak C^*-Hopf algebra with the Haar projection e_2H and normalized Haar functional $\tilde{\phi}(b)=\phi(H\tilde{S}(H)\,b)=d\tau(\tilde{S}(H)\,Hb)$ (cf. Theorem* 4.17).

Proof. It follows from Propositions 5.3–5.6 that $(B,\tilde{\Delta},\tilde{\varepsilon},\tilde{S})$ is a weak C^*-Hopf algebra. The properties of e_2 established in Theorem 4.17 and Proposition 4.2 give

$$be_2H=\varepsilon_B^t(b)\,e_2H=\tilde{\varepsilon}^t(b)\,e_2H,$$
$$\tilde{\varepsilon}^t(e_2H)=\varepsilon_B^t(e_2H)=\lambda^{-1}E_{M_1}(e_2He_2)$$
$$=\lambda^{-1}E_{M_1}(E_M(H)\,e_2)=E_M(H)=1,$$

since $\tilde{\varepsilon}^t=\varepsilon_B^t$ by Proposition 5.5, and $E_M(H)=\tau(H)\,1=1$ by Corollary 4.7.

Using Lemma 5.2(ii), and taking into account that $\tilde{S}^2|_{B_t}=\mathrm{id}_{B_t}$ (Proposition 5.6), we compute for all $b\in B$, $z\in B_t$:

$$\tilde{\varepsilon}^t(\tilde{S}(z))=\tilde{S}(z)_{(\tilde{1})}\,\tilde{S}(\tilde{S}(z)_{(\tilde{2})})=1_{(\tilde{1})}\tilde{S}(\tilde{S}(z)\,1_{(\tilde{2})})=\tilde{S}^2(z)=z,$$

therefore $e_2\tilde{S}(H)=e_2\tilde{\varepsilon}^t(\tilde{S}(H))^*=e_2H$. Since $S_B(e_2)=e_2$ and $S_B(H)=\tilde{S}(H)$, using the above relation, we get

$$\tilde{S}(e_2H)=\tilde{S}(H)^{-1}\,S_B(e_2H)\,\tilde{S}(H)=e_2\tilde{S}(H)=e_2H.$$

Thus $\tilde{S}(e_2H)=e_2H$. Also we have:

$$(e_2H)^2=E_M(H)\,e_2H=e_2H,$$
$$(e_2H)^\dagger=\tilde{S}(H)^{-1}\,He_2\tilde{S}(H)=e_2\tilde{S}(H)=e_2H.$$

Therefore, e_2H is an $\tilde{S}$-invariant projection. This proves that e_2 is the Haar projection of B.

Next, using Lemma 5.2 and the properties of the trace ϕ from the proof of Theorem 4.17 we have

$$\begin{aligned}
\tilde{\varepsilon}^t(b_{(\tilde{1})})\,\tilde{\phi}(b_{(\tilde{2})}) &= \varepsilon^t_B(b_{(1)})\,\tilde{\phi}(H^{-1}b_{(2)}) = \varepsilon^t_B(b_{(1)})\,\phi(\tilde{S}(H)\,b_{(2)})\\
&= \varepsilon^t_B((b\tilde{S}(H))_{(1)})\,\phi((b\tilde{S}(H))_{(2)}) = (b\tilde{S}(H)))_{(1)}\,\phi((b\tilde{S}(H))_{(2)})\\
&= b_{(1)}\phi(\tilde{S}(H)\,b_{(2)}) = b_{(1)}\phi(H\tilde{S}(H)\,H^{-1}b_{(2)})\\
&= b_{(1)}\tilde{\phi}(H^{-1}b_{(2)}) = b_{(\tilde{1})}\tilde{\phi}(b_{(\tilde{2})}),\\
\tilde{\phi}(\tilde{S}(b)) &= \phi(H\tilde{S}(H)\,\tilde{S}(H)^{-1}\,S_B(b)\,\tilde{S}(H)) = \phi(H\tilde{S}(H)\,S_B(b))\\
&= \phi(b\tilde{S}(H)\,H) = \tilde{\phi}(b),\\
\tilde{\phi}(\tilde{\varepsilon}^t(b)) &= \phi(\tilde{S}(H)\,H\varepsilon^t_b(b)) = \tau(\tilde{S}(H))\,\phi(\varepsilon^t_B(Hb)) = \varepsilon_B(Hb) = \tilde{\varepsilon}(b),
\end{aligned}$$

therefore, $\tilde{\phi}$ is the normalized Haar functional on B.

Remark 5.8. (i) The non-degenerate duality $\langle\ ,\ \rangle$ induces on $A = N' \cap M_1$ the structure of the weak C^*-Hopf algebra dual to B.

(ii) The weak C^*-Hopf algebra B is biconnected, since the inclusion $B_t = M' \cap M_1 \subset B = M' \cap M_2$ is connected ([9], 4.6.3) and $B_t \cap B_s = (M' \cap M_1) \cap (M'_1 \cap M_2) = \mathbb{C}$. Thus, only biconnected weak Hopf C^*-algebras arise as symmetries of finite index depth 2 inclusions of II_1 factors.

(iii) The principal graph of the inclusion $M \subset M_1$ is given by the Bratteli diagram of $B_t \subset B$. The index is the square of the norm of the inclusion matrix Λ of $B_t \subset B$:

$$\lambda^{-1} = [M : N] = [M_1 : M] = \|\Lambda\|^2,$$

so we can call this number the *index* of B.

(iv) If λ^{-1} is not integer, then $\tilde{S}$ has infinite order. Indeed, the canonical element G implementing the square of the antipode in Proposition 5.6 is positive, so if $\tilde{S}^{2n} = \mathrm{id}$ for some n, then G^n belongs to $Z(B)$, the center of B. Taking the nth root, we get that $G \in Z(B)$, which means that $S^2 = \mathrm{id}$, and B is a weak Kac algebra, which is in contradiction with Theorem 4.17.

6. ACTION OF B ON M_1

Note that in terms of $\tilde{\Delta}$, Proposition 4.13 means that

$$E_{M_1}(bxye_2) = \lambda^{-1} E_{M_1}(b_{(\tilde{1})}xe_2)\,E_{M_1}(b_{(\tilde{2})}ye_2),$$

for all $b \in B$ and $x, y \in M_1$. This suggests the following definition of the action of B on M_1.

PROPOSITION 6.1. *The map* $\triangleright : B \otimes M_1 \to M_1$:

$$b \triangleright x = \lambda^{-1} E_{M_1}(bxe_2)$$

defines a left action of B *on* M_1 (*cf.* [23, *Proposition* 17]).

Proof. Clearly, the above map defines a left B-module structure on M_1, since $1 \triangleright x = x$ and

$$b \triangleright (c \triangleright x) = \lambda^{-2} E_{M_1}(bE_{M_1}(cxe_2)\, e_2) = \lambda^{-1} E_{M_1}(bcxe_2) = (bc) \triangleright x.$$

Next, using Proposition 4.13 we get

$$\begin{aligned} b \triangleright xy &= \lambda^{-1} E_{M_1}(bxye_2) = \lambda^{-2} E_{M_1}(b_{(\tilde{1})} xe_2)\, E_{M_1}(b_{(\tilde{2})} ye_2) \\ &= (b_{(\tilde{1})} \triangleright x)(b_{(\tilde{2})} \triangleright y). \end{aligned}$$

By Remark 4.4 and properties of S_B we also get

$$\begin{aligned} \tilde{S}(b)^\dagger \triangleright x^* &= \lambda^{-1} E_{M_1}(\tilde{S}(b)^\dagger\, x^* e_2) \\ &= \lambda^{-1} E_{M_1}(S_B(H)^{-1}\, S_B(HbH^{-1})^*\, S_B(H)\, x^* e_2) \\ &= \lambda^{-1} E_{M_1}(S_B(b^*)\, x^* e_2) \\ &= \lambda^{-1} E_{M_1}(e_2 x^* b^*) = \lambda^{-1} E_{M_1}(bxe_2)^* = (b \triangleright x)^*. \end{aligned}$$

Finally,

$$b \triangleright 1 = \lambda^{-1} E_{M_1}(be_2) = \lambda^{-1} E_{M_1}(\lambda^{-1} E_{M_1}(be_2)\, e_2) = \tilde{\varepsilon}^t(b) \triangleright 1,$$

and $b \triangleright 1 = 0$ iff $\tilde{\varepsilon}^t(b) = \lambda^{-1}(be_2) = 0$.

PROPOSITION 6.2. $M_1^B = M$, *i.e.*, M *is the fixed point subalgebra of* M_1.

Proof. If $x \in M_1$ is such that $b \triangleright x = \tilde{\varepsilon}^t(b) \triangleright x$ for all $b \in B$, then $E_{M_1}(bxe_2) = E_{M_1}(\varepsilon_B^t(b)\, xe_2) = E_{M_1}(be_2)\, x$. Taking $b = e_2$, we get $E_M(x) = x$ which means that $x \in M$. Thus, $M_1^B \subset M$.

Conversely, if $x \in M$, then x commutes with e_2 and

$$b \triangleright x = \lambda^{-1} E_{M_1}(be_2 x) = \lambda^{-1} E_{M_1}(\lambda^{-1} E_{M_1}(be_2)\, e_2 x) = \varepsilon_B^t(b) \triangleright x,$$

therefore $M_1^B = M$.

PROPOSITION 6.3. *The map* $\theta : [x \otimes b] \mapsto x\tilde{S}(H)^{1/2}\, b\tilde{S}(H)^{-1/2}$ *defines a von Neumann algebra isomorphism between* $M_1 \rtimes B$ *and* M_2.

Proof. By definition of the action $\triangleright$ we have:

$$\theta([x(z \triangleright 1) \otimes b]) = x\tilde{S}(H)^{1/2}\, \lambda^{-1} E_{M_1}(ze_2)\, b\tilde{S}(H)^{-1/2}$$
$$= x\tilde{S}(H)^{1/2}\, zb\tilde{S}(H)^{-1/2} = \theta([x \otimes zb]),$$

for all $x \in M_1$, $b \in B$, $z \in B_t$, so θ is a well defined linear map from $M_1 \rtimes B = M_1 \otimes_{B_t} B$ to M_2. It is surjective since an orthonormal basis of $B = M' \cap M_2$ over $B_t = M' \cap M_1$ is also a basis of M_2 over M_1 ([21], 2.1.3).

Let us check that θ is an involution-preserving isomorphism of algebras. Note that from Corollary 4.12 we have $bx = (b_{(\tilde{1})} \triangleright x)\, b_{(\tilde{2})}$. This allows us to compute, for all $x, y \in M_1$ and $b, c \in B$:

$$\theta([x \otimes b][y \otimes c]) = \theta([x(b_{(\tilde{1})} \triangleright y) \otimes b_{(\tilde{2})} c])$$
$$= x(b_{(\tilde{1})} \triangleright y)\, \tilde{S}(H)^{1/2}\, b_{\tilde{2})} c\tilde{S}(H)^{-1/2}$$
$$= x((\tilde{S}(H)^{1/2}\, b)_{(\tilde{1})} \triangleright y)(\tilde{S}(H)^{1/2}\, b)_{(\tilde{2})}\, c\tilde{S}(H)^{-1/2}$$
$$= x\tilde{S}(H)^{1/2}\, byc\tilde{S}(H)^{-1/2}$$
$$= x\tilde{S}(H)^{1/2}\, b\tilde{S}(H)^{-1/2}\, y\tilde{S}(H)^{1/2}\, c\tilde{S}(H)^{-1/2}$$
$$= \theta([x \otimes b])\, \theta([y \otimes c]),$$
$$\theta([x \otimes b]^*) = (b^{\dagger}_{(1)} \triangleright x^*)\, \tilde{S}(H)^{1/2}\, b^{\dagger}_{(2)} \tilde{S}(H)^{-1/2}$$
$$= (\tilde{S}(H)^{1/2}\, b^{\dagger})_{(\tilde{1})} \triangleright x^*)(\tilde{S}(H)^{1/2}\, b^{\dagger})_{(\tilde{2})}\, \tilde{S}(H)^{-1/2}$$
$$= \tilde{S}(H)^{1/2}\, b^{\dagger} \tilde{S}(H)^{-1/2}\, x^*$$
$$= \tilde{S}(H)^{-1/2}\, b^* \tilde{S}(H)^{1/2}\, x^*$$
$$= (x\tilde{S}(H)^{1/2}\, b\tilde{S}(H)^{-1/2})^*$$
$$= \theta([x \otimes b])^*.$$

It is known that $M_1 \rtimes B$ is a II_1 factor iff M_1^B is [17]. Now the injectivity of θ follows from the simplicity of II_1 factors (see, e.g., the appendix of [11]). Thus, θ is a von Neumann algebra isomorphism.

Remark 6.4. (i) The action of B constructed in Proposition 6.1 is minimal, since we have $M_1' \cap M_1 \rtimes B = M_1' \cap M_2 = B_s$ by Proposition 6.3.

(ii) If the inclusion $N \subset M$ is irreducible, then B is a usual Kac algebra (i.e., a Hopf C^*-algebra) and we recover the well-known result proved in [23, 13], and [4].

7. EXAMPLES

Depth 2 Subfactors and Weak Kac Algebras of Index 4

It is known ([21], Section 5) that there are exactly 3 non-isomorphic reducible subfactors of depth 2 and index 4 of the hyperfinite II_1 factor R. One of them has the principal graph ([9], Definition 4.6.5) $A_1^{(1)}$ (this one is, of course, $R \subset R \otimes M_2(\mathbb{C})$), other two have the principal graph $A_3^{(1)}$ (they can be viewed as diagonal subfactors of the form $\{(\begin{smallmatrix} x & 0 \\ 0 & \alpha(x) \end{smallmatrix}) \mid x \in R\}$ of $R \otimes M_2(\mathbb{C})$, where α is an outer automorphism of R such that $\alpha^2 = \mathrm{id}$, resp. α^2 is inner and $\alpha^2 \neq \mathrm{id}$). The graphs $A_1^{(1)}$ and $A_3^{(1)}$ are pictured on Fig. 1.

Since we have shown that there is a bijective correspondence between depth 2 subfactors and biconnected weak C^*-Hopf algebras, this means that there exist exactly 3 non-isomorphic biconnected weak C^*-Hopf algebras (which are not usual Hopf algebras) of index 4. Let us describe them.

Two biconnected weak Kac algebras of dimensions 8 and 16 with the principal graphs $A_3^{(1)}$ and $A_1^{(1)}$ respectively were constructed in ([14], 3.1, 3.2). Let us show how the remaining weak Kac algebra with the principal graph $A_3^{(1)}$ can be obtained as a twisting, following the line of [6, 24, 15].

Let $(B, \Delta, S, \varepsilon)$ be a weak Kac algebra, $\Omega \in B \otimes B$ a partial isometry such that $\Omega^*\Omega = \Omega\Omega^* = \mathscr{E} = \Delta(1)$. A *2-coboundary* $\partial_2 \Omega \in B \otimes B \otimes B$ is defined by:

$$\partial_2 \Omega = (\mathrm{id} \otimes \Delta)(\Omega^*)(1 \otimes \Omega^*)(\Omega \otimes 1)(\Delta \otimes \mathrm{id})(\Omega).$$

A 2-*pseudo-cocycle* (resp., 2-*cocycle*) on B is such an Ω that $\partial_2 \Omega$ commutes with $(\Delta \otimes \mathrm{id})\, \Delta(B)$ (resp, $\partial_2 \Omega = ((\mathscr{E} \otimes 1)(1 \otimes \mathscr{E}))$—(cf. [24], 2.2). A 2-pseudo-cocycle is said to be *counital* if $(\varepsilon \otimes \mathrm{id})(\Omega \mathscr{E}) = (\mathrm{id} \otimes \varepsilon)(\Omega \mathscr{E}) = 1$.

Let us put, for all x in M, $\Delta_\Omega(x) = \Omega \Delta(x)\, \Omega^*$. One can show as in ([24], 2.3) that Δ_Ω is coassociative iff Ω is a 2-pseudo-cocycle on B. Let Ω be a 2-pseudo-cocycle on B and $u \in B$ be a unitary such that $(u \otimes u)\, \mathscr{E} = \mathscr{E}(u \otimes u)$. Then one can show as in ([24], 2.8) that $S_u(x) = uS(x)\, u^*$ is anti-comultiplicative with respect to Δ_Ω iff $\varsigma(S \otimes S)(\Omega)\, \Omega^u$ commutes with

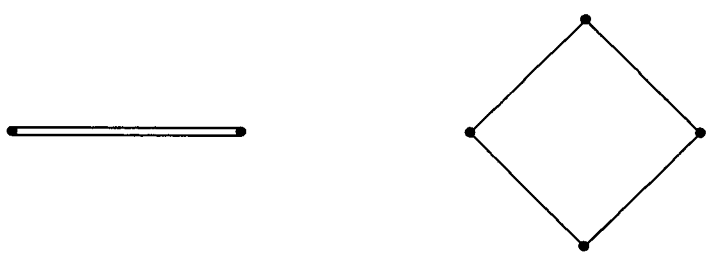


FIG. 1. The graphs $A_1^{(1)}$ and $A_3^{(1)}$.

$\Delta(B)$, where $\Omega^u=(u^*\otimes u^*)\,\Omega\Delta(u)$, ς is the usual flip in $B\otimes B$. Such an Ω is said to be *pseudo-coinvolutive* (cf. [24], 2.10).

The following *lifting* approach (see [6, 24]) gives a way of constructing concrete Ω and u. Suppose that there exists a groupoid G with the unit space G^0, such that $\mathbb{C}(G)$, the algebra of functions on G ([14], 2.1.4(b)), is a weak Kac subalgebra of B. Then one can construct Ω and u as follows:

$$\Omega=\sum_{x,\,y\in G}\omega(x,y)(P_x\otimes P_y),\qquad u=\mu(\mathrm{id}\otimes S)(\Omega)=\sum_{x\in G}\omega(x,x^{-1})\,P_x,$$

where P_x $(x\in G)$ is a system of mutually orthogonal projections generating $\mathbb{C}(G)$, $\mu\colon B\otimes B\to B$ is a multiplication, x^{-1} is the inverse of $x\in G$, and ω is a complex function on $G\times G$. One can see that Ω is counital iff

$$\omega(x,y)=\delta_{s(x),\,y},\ \omega(y,x)=\delta_{t(x),\,y}\qquad \forall x\in G,\ y\in G^0$$

and u is unitary iff $|\omega(x,x^{-1})|=1\ \forall x\in G$.

LEMMA 7.1. *Let in the above situation Ω be a pseudo-coinvolutive 2-pseudo-cocycle lifted from a weak Kac subalgebra $\mathbb{C}(G)\subset B$ such that u satisfies the property $u^*S(u)\in Z(B)$. Then the twisted algebra $(B,\Delta_\Omega,S_u,\varepsilon)$ is a weak Kac algebra with the same Cartan subalgebras B_s and B_t.*

Proof. Let us show that

$$\tilde\varepsilon_t=\mu(\mathrm{id}\otimes S_u)\,\Delta_\Omega=\varepsilon_t,\qquad \tilde\varepsilon_s=\mu(S_u\otimes\mathrm{id})\,\Delta_\Omega=\varepsilon_s.$$

Indeed, since $\mathbb{C}(G)$ is commutative and contains B_s and B_t ([14], 2.1.12), we have

$$\begin{aligned}\tilde\varepsilon_t(x)&=\Omega^{(1)}x_{(1)}\Omega^{(1)*}uS(\Omega^{(2)}x_{(2)}\Omega^{(2)*})\,u^*\\&=\Omega^{(1)}x_{(1)}\Omega^{(1)*}\Omega'^{(1)}S(\Omega'^{(2)})\,S(\Omega^{(2)*})\,S(x_{(2)})\,S(\Omega^{(2)})\,u^*\\&=\Omega^{(1)}x_{(1)}S(x_{(2)})\,S(\Omega^{(2)})\,u^*=\varepsilon_t(x)\,uu^*=\varepsilon_t(x),\end{aligned}$$

where Ω' stands for another copy of Ω. Similarly for $\tilde\varepsilon_s$, using the property $u^*S(u)\in Z(B)$.

Now it is clear that $\mathrm{Im}\,\tilde\varepsilon_t=\mathrm{Im}\,\varepsilon_t=B_t$, $\mathrm{Im}\,\tilde\varepsilon_s=\mathrm{Im}\,\varepsilon_s=B_s$, and according to Theorem 2.6.1 of [14], it is enough to show that $(\varepsilon\otimes\mathrm{id})\,\Delta_\Omega=\mathrm{id}=(\mathrm{id}\otimes\varepsilon)\,\Delta_\Omega$. Let us prove, for instance, the first equality. The axiom (2) of the definition of weak Kac algebra (see Preliminaries) and counitality of Ω imply:

$$(\varepsilon\otimes\mathrm{id})(\Omega\Delta(x))=(\varepsilon\otimes\mathrm{id})(\Omega\mathscr{E}(1\otimes x))=x,$$

from where we also have $(\varepsilon \otimes \mathrm{id})(\Delta(x)\,\Omega^*) = x (\forall x \in B)$. Next, as an easy consequence of the axiom (2), we get (cf. [2]):

$$\varepsilon(xyz) = \varepsilon(xy_{(1)})\,\varepsilon(y_{(2)}z) \qquad (\forall x, y, z \in B).$$

Finally, the above relations imply

$$\begin{aligned}(\varepsilon \otimes \mathrm{id})(\Omega\Delta(x)\,\Omega^*) &= \varepsilon(\Omega^{(1)}x_{(1)})\,\Omega^{(2)}\varepsilon(x_{(2)}\Omega^{*(1)})\,x_{(3)}\Omega^{*(2)}\\ &= \varepsilon(\Omega^{(1)}x_{(1)})\,\Omega^{(2)}x_{(2)} = x.\end{aligned}$$

EXAMPLE 7.2. Let us apply the above construction to the biconnected weak Kac algebra $B = \mathbb{C}(M_2) \rtimes \mathbb{Z}/2\mathbb{Z} \cong M_2(\mathbb{C}) \oplus M_2(\mathbb{C})$ described in ([14], 3.1), where the right action of $\mathbb{Z}/2\mathbb{Z} = \{\mathrm{id}, \sigma\}$ on the commutative weak Kac algebra $\mathbb{C}(M_2) = \mathrm{span}\{e_{ij}\}_{i,j=1}^{2}$ of functions on the transitive principal groupoid M_2 on a set of 2 elements is given by $e_{ij} \triangleleft \sigma = e_{\sigma(i)\,\sigma(j)}$, where $\sigma(1) = 2$, $\sigma(2) = 1$.

The maps Δ, S, ε defining the weak Kac algebra structure on B can be described as follows:

$$\Delta(e_{ij} \otimes \gamma) = (e_{i1} \otimes \gamma) \otimes (e_{1j} \otimes \gamma) + (e_{i2} \otimes \gamma) \otimes (e_{2j} \otimes \gamma),$$

$$S(e_{ij} \otimes \gamma) = e_{\gamma(j)\,\gamma(i)} \otimes \gamma^{-1}, \qquad \varepsilon(e_{ij} \otimes \gamma) = \delta_{ij},$$

for all $e_{ij} \in \mathbb{C}(M_2)$, $\gamma \in \mathbb{Z}/2\mathbb{Z}$.

To construct a counital 2-pseudo-cocycle we must have a function $\omega\colon M_2 \times M_2 \to \mathbb{C}$ such that $\omega(e_{ii}, e_{kl}) = \omega(e_{lk}, e_{ii}) = \delta_{ik}$ $(i, k, l = 1, 2)$. Also, let us define $\omega(e_{21}, e_{12}) = 1$, $\omega(e_{12}, e_{21}) = -1$, $\omega(e_{21}, e_{21}) = \omega(e_{12}, e_{12}) = 0$. Then ω is completely determined and we have $\Omega = \mathscr{E} - 2e_{12} \otimes e_{21}$. Clearly, $\Omega^*\Omega = \Omega\Omega^* = \mathscr{E}$, and

$$\partial_2\Omega = (\mathscr{E} \otimes 1)(1 \otimes \mathscr{E}) - 2(e_{21} \otimes e_{12} \otimes e_{21} + e_{12} \otimes e_{21} \otimes e_{12})$$

which commutes with $(\Delta \otimes \mathrm{id})\,\Delta(B)$, so Ω is a 2-pseudo-cocycle. Then the unitary $u = \mu(\mathrm{id} \otimes S)\,\Omega = 1 - 2e_{12}$ satisfies the property $u^*S(u) \in Z(B)$, and $\varsigma(S \otimes S)(\Omega)\,\Omega^u = \mathscr{E} - 2(e_{21} \otimes e_{12} + e_{12} \otimes e_{21})$ commutes with $\Delta(B)$. Thus, according to Lemma 7.1, $(B, \Delta_\Omega, S_u, \varepsilon)$ is also a weak Kac algebra with the same Cartan subalgebras. As for explicit formulae for Δ_Ω and S_u, one can check that Δ_Ω differs from Δ only on $e_{ii} \otimes \sigma$ $(i = 1, 2)$:

$$\Delta_\Omega(e_{ii} \otimes \sigma) = (e_{ii} \otimes \sigma) \otimes (e_{ii} \otimes \sigma) - (e_{i\sigma(i)} \otimes \sigma) \otimes (e_{\sigma(i)\,i} \otimes \sigma)$$

and S_u differs from S only on $e_{i\sigma(i)} \otimes \sigma$:

$$S_u(e_{i\sigma(i)} \otimes \sigma) = -e_{i\sigma(i)} \otimes \sigma.$$

One can easily verify that this weak Kac algebra is biconnected and not isomorphic to the initial one, because the eigenspace corresponding to eigenvalue -1 of the antipode S (resp., S_u), viewed as a linear transformation of the vector space B, is 2-dimensional (resp., 4-dimensional).

Weak C^*-Hopf Algebra Arising From the Golden Ratio Subfactor

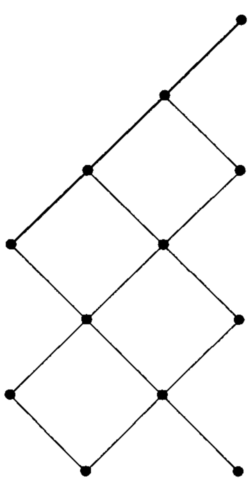
Let us note that weak C^*-Hopf algebras can be canonically constructed from II_1 subfactors of *any* finite depth. Namely, if $N \subset M$ is a finite index inclusion of II_1 factors of depth $n \geqslant 2$ with the corresponding Jones tower $N \subset M \subset M_1 \subset M_2 \cdots$, then the inclusion $N \subset M_{n-2}$ is of depth 2.

Indeed, it follows from [20] that

$$N \subset M_{n-2} \subset M_{2n-3} \subset M_{3n-4} \cdots$$

is the Jones tower for the inclusion $N \subset M_{n-2}$. The depth n condition for $N \subset M$ implies that $\dim Z(N' \cap M_{n-2}) = \dim Z(N' \cap M_{3n-4})$, therefore the inclusion $N \subset M_{n-2}$ is of depth 2, and according to Theorem 5.7 and Remark 5.8(i), there is a structure of a weak C^*-Hopf algebra on $N' \cap M_{2n-3}$.

EXAMPLE 7.3. Let us compute the structure of the weak C^*-Hopf algebra associated with the A_3 subfactor ([10], 5.2), i.e., the unique subfactor with index $\lambda^{-1} = 4\cos^2(\pi/5) = \phi^2$, $\phi =$ golden ratio (Fig. 2).



$N' \cap N \cong \mathbb{C}$

$N' \cap M \cong \mathbb{C}$

$N' \cap M_1 \cong \mathbb{C} \oplus \mathbb{C}$

$N' \cap M_2 \cong \mathbb{C} \oplus M_2(\mathbb{C})$

$N' \cap M_3 \cong M_3(\mathbb{C}) \oplus M_2(\mathbb{C})$

$N' \cap M_4 \cong M_3(\mathbb{C}) \oplus M_5(\mathbb{C})$

$N' \cap M_5 \cong M_8(\mathbb{C}) \oplus M_5(\mathbb{C})$

FIG. 2. The tower of relative commutants of the A_3 subfactor.

In this case the depth is $n=3$, so using the above observation we get that $A=N'\cap M_3$ and $B=M_1'\cap M_5$ are weak C^*-Hopf algebras dual to each other. Note that $A=\{1, e_1, e_2, e_3\}''$ and $B=\{1, e_3, e_4, e_5\}''$ where e_i, $i=1, 2, \ldots$ are the Jones projections of the tower, $M_i=\{M_{i-1}, e_i\}''$, see ([9], 4.7b). By [20], the Jones projections implementing the conditional expectations from M_3 to M_1 and from M_5 to M_3 are $f_1=\lambda^{-1}(e_2e_1)(e_3e_2)$ and $f_2=\lambda^{-1}(e_4e_3)(e_5e_4)$ respectively. Thus, the duality form between A and B (Proposition 3.2) is given by

$$\begin{aligned}\langle a, b\rangle &= \dim(N'\cap M_1)[M_1 : N]^2\, \tau(af_2f_1b)\\ &= 2\lambda^{-6}\tau(ae_4e_3e_2e_1e_5e_4e_3e_2b),\end{aligned}$$

for all $a\in A$, $b\in B$. Using this form and the commutation relations between the Jones projections e_i, $i=1, 2, \ldots$, we can explicitly write down the weak C^*-Hopf algebra structure of A, repeating the general construction of Definition 3.3 and deformation procedure of Definition 5.1.

As an algebra, A is isomorphic to $M_2(\mathbb{C})\oplus M_3(\mathbb{C})$, the source and target Cartan subalgebras are isomorphic to $\mathbb{C}\oplus\mathbb{C}$, we have $A_s=\mathbb{C}e_3\oplus\mathbb{C}(1-e_3)$ and $A_t=\mathbb{C}e_1\oplus\mathbb{C}(1-e_1)$. Note that the canonical element H implementing the deformation (Corollary 4.7) is $H=\frac{1}{2}(\lambda^{-1}e_3+(1-\lambda)^{-1}(1-e_3))$, so the involution on the generating idempotents is given by

$$\begin{aligned}&1^\dagger=1, \qquad e_1^\dagger=e_1, \qquad e_3^\dagger=e_3,\\ &e_2^\dagger=(\lambda e_3+(1-\lambda)(1-e_3))\, e_2(\lambda e_3+(1-\lambda)(1-e_3))^{-1}.\end{aligned}$$

The formulas for the comultiplication are

$$\begin{aligned}\tilde{\Delta}(1)&=e_3\otimes e_1+(1-e_3)\otimes(1-e_1),\\ \tilde{\Delta}(e_1)&=e_1e_3\otimes e_1+e_1(1-e_3)\otimes(1-e_1),\\ \tilde{\Delta}(e_3)&=e_3\otimes e_1e_3+(1-e_3)\otimes(1-e_1)\,e_3,\\ \tilde{\Delta}(e_2)&=\left(1-\frac{(e_3-e_2)^2}{1-\lambda}\right)\otimes\left(1-\frac{(e_1-e_2)^2}{1-\lambda}\right)+\lambda e_3\otimes e_1\\ &\quad+(1-\lambda)\,\frac{\lambda e_3-e_2e_3}{\sqrt{\lambda-\lambda^2}}\otimes\frac{\lambda e_1-e_2e_1}{\sqrt{\lambda-\lambda^2}}+\lambda\,\frac{\lambda e_3-e_3e_2}{\sqrt{\lambda-\lambda^2}}\otimes\frac{\lambda e_1-e_1e_2}{\sqrt{\lambda-\lambda^2}}\\ &\quad+(1-\lambda)\left(\frac{(e_3-e_2)^2}{1-\lambda}-e_3\right)\otimes\left(\frac{(e_1-e_2)^2}{1-\lambda}-e_1\right).\end{aligned}$$

Note that left (resp., right) tensor factors in the last formula form a system of matrix units in $M'\cap M_3$ (resp., $N'\cap M_2$).

The counit is completely determined by its values on the reduced words of e_i's:

$$\tilde{\varepsilon}(1)=\tilde{\varepsilon}(e_2)=2, \qquad \tilde{\varepsilon}(e_1)=\tilde{\varepsilon}(e_3)=\tilde{\varepsilon}(e_1e_3)=1,$$
$$\tilde{\varepsilon}(e_1e_2)=\tilde{\varepsilon}(e_3e_2)=\tilde{\varepsilon}(e_1e_3e_2)=1, \qquad \tilde{\varepsilon}(e_1e_2e_3)=\tilde{\varepsilon}(e_3e_2e_1)=\lambda,$$
$$\tilde{\varepsilon}(e_2e_1)=\tilde{\varepsilon}(e_2e_3)=\tilde{\varepsilon}(e_2e_1e_3)=\tilde{\varepsilon}(e_2e_1e_3e_2)=2\lambda.$$

Finally, the antipode is determined by

$$\tilde{S}(1)=1, \qquad \tilde{S}(e_1)=e_3, \qquad \tilde{S}(e_3)=e_1,$$
$$\tilde{S}(e_2)=(\lambda e_3+(1-\lambda)(1-e_3))\, e_2(\lambda e_3+(1-\lambda)(1-e_3))^{-1}.$$

Thus, $M_2(\mathbb{C})\oplus M_3(\mathbb{C})$ has a structure of a weak C^*-Hopf algebra (which is not a weak Kac algebra) with the index $\lambda^{-2}=16\cos^4(\pi/5)$. Note that an example of a weak C^*-Hopf algebra with the same algebra structure and principal graph was constructed in [3] without using subfactors.

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